Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 13, NUMBER 2 (2010) ISSN 1311-0454

SOLUTION OF SPACE-TIME FRACTIONAL SCHRÖDINGER EQUATION OCCURRING IN QUANTUM MECHANICS

R.K. Saxena [a], **Ravi Saxena** [b] **and S.L. Kalla** [c]

Dedicated to Professor A.M. Mathai
on the occasion of his 75^{th} *birthday*

Abstract

The object of this article is to present the computational solution of one-dimensional space-time fractional Schrödinger equation occurring in quantum mechanics. The method followed in deriving the solution is that of joint Laplace and Fourier transforms. The solution is derived in a closed and computational form in terms of the H-function. It provides an elegant extension of a result given earlier by Debnath, and by Saxena et al. The main result is obtained in the form of Theorem 1. Three special cases of this theorem are given as corollaries. Computational representation of the fundamental solution of the proposed equation is also investigated.

Mathematics Subject Classification 2010: 26A33, 44A10, 33C60, 35J10

Key Words and Phrases: Mittag-Leffler function, quantum mechanics, Fourier transform, H-function, fractional Schrödinger equation, Caputo derivative, Lévy distribution, Feynman path integrals and Gaussian probability distribution

---

1. Introduction

Feynman and Hibbs [8] reconstructed the Schrödinger equation by making use of the path integral approach by considering a Gaussian probability distribution. This approach is further extended by Laskin [16, 17, 18] in formulating the fractional Schrödinger equation by generalizing the Feynman path integrals from Brownian-like to Levy-like quantum mechanical paths. The Schrödinger equation thus obtained contains space and time fractional derivatives. In a similar manner, one obtains a time fractional equation if non-Marcovian evolution is considered. In a recent paper, Naber [22] discussed certain properties of time fractional Schrödinger equation by writing the Schrdinger equation in terms of fractional derivatives as dimensionless objects. Time fractional Schrödinger equations are also discussed by Debnath [4], Bhatti [1], and Debnath and Bhatti [6].

In a recent paper [29], the authors investigated the solution of the following generalized one dimensional fractional Schrödinger equation of a free particle of mass m defined by

$$\frac{\partial^\alpha N}{\partial t^\alpha} = (i\hbar/2m)\frac{\partial^\beta}{\partial x^\beta}N(x,t), \quad -\infty<x<\infty, t>0,\ 0<\alpha\leq 1,\ \beta>0, \quad (1.1)$$

$$N(x,0) = N_0(x), \quad -\infty < x < \infty, \quad (1.2)$$

$$N(x,t) \to 0 \quad \text{as} \quad |x| \to \infty, \quad (1.3)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative defined by (2.7) and $\frac{\partial^\beta}{\partial x^\beta}$ is the Liouville fractional space derivative [26], $N(x,t)$ is the wave function,

$$h = 2\pi\hbar = 6.625 \times 10^{-27} erg\sec = 4.14 \times 10^{-21} MeV\sec. \quad (1.4)$$

is the Planck constant and $N_0(x)$ is an arbitrary function.

The probability structure of time fractional Schrödinger equation is discussed by Tofight [31]. Some physical applications of fractional Schrödinger equation are investigated by Guo and Xu [9] by deriving the solution for a free particle and an infinite square potential well. This has motivated the authors to investigate the solution of one-dimensional space-time fractional generalization of the Schrödinger equation (3.1) occurring in quantum mechanics containing a fractional generalization of the ordinary Laplace operator $\Delta^{\alpha/2}$, defined by (2.14).

Fractional reaction-diffusion equations are solved by Haubold et al. [10], Saxena et al. [26, 27, 28], and Henry and Wearne [11].

2. Mathematical prerequsites

The definitions of the well-known Laplace and Fourier transforms of a function $N(x,t)$ and their inverses are described below:

The Laplace transform of a function $N(x,t)$ [continuous or partially continuous and of exponential order as t approaches infinity] with respect to t is defined by

$$L\{N(x,t)\} = N^{\sim}(x,s) = \int_0^\infty e^{-st} N(x,t)dt, \qquad (t > 0, \quad x \in \mathbb{R}) \quad (2.1)$$

where $\Re(s) > 0$, and its inverse transform with respect to s is given by

$$L^{-1}\{N^{\sim}(x,s)\} = N(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} N(x,s)ds, \quad (2.2)$$

γ being a fixed real number.

The Fourier transform of a function $N(x,t)$ with respect to x is defined by

$$F\{N(x,t)\} = N^*(k,t) = \int_{-\infty}^{\infty} e^{ikx} N(x,t)dx, \qquad (k > 0), \quad (2.3)$$

and the inverse Fourier transform with respect to k is given by the formula

$$F^{-1}\{N^*(k,t)\} = N(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} N^*(k,t)dk. \quad (2.4)$$

The space of functions for which the transforms defined by (2.1) and (2.3) exist is denoted by

$$LF = L(\mathbb{R}_+) \times F(\mathbb{R}).$$

The right-sided Riemann-Liouville fractional integral of order ν is defined by Miller and Ross [21, p. 45], Samko et al. [25]:

$${}_a^{RL}D_t^{-\nu} N(x,t) = \frac{1}{\Gamma(\nu)}\int_a^t (t-u)^{\nu-1} N(x,u)du, \qquad (t > a), \quad (2.5)$$

where $Re(\nu) > 0$. The right-sided Riemann-Liouville fractional derivative of order μ is defined as

$${}_a^{RL}D_t^{\mu} N(x,t) = \left(\frac{d}{dx}\right)^n (I_a^{n-\mu} N(x,t)) \quad (Re(\mu) > 0,\ n = |Re(\mu)|+1), \quad (2.6)$$

where $[x]$ represents the integral part of the number x.

The following fractional derivative of order $\alpha > 0$ is introduced by Caputo [3] in the form (if $m-1 < \alpha \leq m$, $Re(\alpha) > 0, m \in \mathbb{N}$):

$$ {}_0^cD_t^\alpha f(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(x,\tau)d\tau}{(t-\tau)^{\alpha+1-m}}\,, \tag{2.7}$$

$$\text{and:} \quad = \frac{\partial^m f(x,t)}{\partial t^m}, \qquad \text{if} \quad \alpha = m. \tag{2.8}$$

where $\frac{\partial^m}{\partial t^m} f$ is the m^{th} partial derivative of order m of the function $f(x,t)$ with respect to t. The Laplace transform of this derivative is given in [15, 19, 20, 21, 24] in the form:

$$L\{{}_0^cD_t^\alpha f(x,t);s\} = s^\alpha F(x,s) - \sum_{r=0}^{m-1} s^{\alpha-r-1} f^{(r)}(x,0+), (m-1 < \alpha \leq m). \tag{2.9}$$

The above formula is useful in deriving the solution of differential and integral equations of fractional order governing certain physical problems of reaction and diffusion. In this connection, one can refer to the monograph by Podlubny [24], Samko et al. [25] and Kilbas et al. [15], Haubold et al. [10], and Saxena et al. [26, 27, 28].

A generalization of the Riemann-Liouville fractional derivative operator (2.6) and Caputo fractional derivative operators (2.7) is given by Hilfer [12], by introducing a right-sided fractional derivative operator of two parameters of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$ in the form

$$ {}_0D_{a+}^{\mu,\nu} N(x,t) = \left(I_{a+}^{\nu(1-\mu)} \frac{\partial}{\partial x} \left(I_{a+}^{(1-\nu)(1-\mu)} N(x,t) \right) \right), \tag{2.10}$$

It is interesting to observe that for $\nu = 0$, (2.10) reduces to the classical Riemann-Liouville fractional derivative operator (2.6). On the other hand, for ν=1 it yields the Caputo fractional derivative operator defined by (2.7). The Laplace transformation formula for this operator is given by Hilfer [12]

$$L\left\{ {}_0D_{0+}^{\mu,\nu} N(x,t);s \right\} = s^\mu N(x,s) - s^{\nu(\mu-1)} I_{0+}^{(1-\nu)(1-\mu)} N(x,0+) \ (0 < \mu < 1), \tag{2.11}$$

where the initial value term

$$I_{0+}^{(1-\nu)(1-\mu)} N(x,0+), \tag{2.12}$$

involves the Riemann-Liouville fractional integral operator of order $(1-\nu)(1-\mu)$ evaluated in the limit as $t \to 0+$. It being understood that the integral

$$N(x,s) = \int_0^\infty e^{-st} N(x,t)dt \tag{2.13}$$

where $\Re(s) > 0$, exists.

In the study of fractional diffusion equations, there often occurs a symmetric fractional generalization of the Laplace operator. Following Brockmann and Sokolev [2, p.419] its one-dimensional variant can be defined as

$$\Delta^{\alpha/2} = \frac{1}{2\cos(\pi\alpha/2)} \left\{ {}_{-\infty}D_x^\alpha + {}_xD_\infty^\alpha \right\}, \quad 0 < \alpha \leq 2, \tag{2.14}$$

where the operators on the right of (2.14) are defined by

$${}_{-\infty}D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^x \frac{f^{(n)}(u)}{(x-u)^{\alpha+1-n}} du, \qquad (n = [\alpha]+1), \tag{2.15}$$

and

$${}_xD_\infty^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^\infty \frac{f^{(n)}(u)}{(u-x)^{\alpha+1-n}} du, \qquad (n = [\alpha]+1), \tag{2.16}$$

The Fourier transform of the operator $\Delta^{\alpha/2}$ is given by [2, p.420]

$$F\{\Delta^{\alpha/2} N(x,t); k\} = -|k|^\alpha N(k,t), \qquad 0 < \alpha \leq 2. \tag{2.17}$$

NOTE 1. The operator, defined by (2.10) also occurs in recent papers by Hilfer [13, 14] and Srivastava et al. [30].

NOTE 2. Applications of fractional calculus in the solution of applied problems can be found in the works [15, 20, 21 23, 24, and 25].

The H-function is defined by means of a Mellin-Barnes type integral in the following manner [26, p.2]:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \left|^{(a_p,A_p)}_{(b_q,B_q)} \right.\right]$$

$$= H_{p,q}^{m,n}\left[z \left|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)} \right.\right] = \frac{1}{2\pi i} \int_L \Theta(\xi) z^{-\xi} d\xi, \tag{2.18}$$

where $i = (-1)^{1/2}$,

$$\Theta(\xi) = \frac{\left[\prod_{j=1}^m \Gamma(b_j + B_j\xi)\right] \left[\prod_{i=1}^n \Gamma(1 - a_j - A_j\xi)\right]}{\left[\prod_{j=m+1}^q \Gamma(1 - b_j - B_j\xi)\right] \left[\prod_{i=n+1}^p \Gamma(a_j + A_j\xi)\right]}, \tag{2.19}$$

and an empty product is always interpreted as unity; $m, n, p, q \in \mathbb{N}_0$ with $0 \leq n \leq p, 1 \leq m \leq q,\ A_i, B_j \in \mathbb{R}_+,\ a_i, b_j \in \mathbb{R}$ or $\mathbb{C}$ $(i = 1, \ldots, p; j = 1, \ldots, q)$ such that

$$A_i(b_j + k) \neq B_j(a_i - \ell - 1), (k, \ell \in N_0; i = 1, \ldots, n; j = 1, \ldots, m), \quad (2.20)$$

where we employ the usual notations: $\mathbb{N}_0 = (0, 1, 2 \ldots)$; $\mathbb{R} = (-\infty, \infty)$; $\mathbb{R}_+ = (0, \infty)$, and $\mathbb{C}$ being the complex number field.

3. Space-time fractional Schrödinger equation

In this section, we will investigate the solution of the one-dimensional space-time fractional Schrödinger equation (3.1). The main result is given in the form of the following Theorem 1.

Theorem 1. *Consider the following one dimensional space-time fractional Schrödinger equation of a free particle of mass m, defined by*

$${}_0D_t^{\mu,\nu} N(x,t) = (\frac{i\hbar}{2m})\Delta^{\alpha/2} N(x,t), \quad 0 < \alpha \leq 2; -\infty < x < \infty, t > 0, \quad (3.1)$$

with initial conditions

$$\left(I_{0+}^{(1-\nu)(1-\mu),0}\right) N(x, 0+) = N_0(x)\ ; \quad -\infty < x < \infty,\ 0 < \mu < 1,\ 0 \leq \nu \leq 1, \quad (3.2)$$

and

$$\lim_{|x|\to\infty} N(x,t) = 0, \quad (3.3)$$

where ${}_0D_t^{\mu,\nu}$ is the generalized Riemann-Liouville fractional derivative operator, defined by (2.10),

$$I_{0+}^{(1-\nu)(1-\mu)} N(x, 0+), \quad (3.4)$$

involves the Riemann-Liouville fractional integral operator of order $(1-\nu)(1-\mu)$ evaluated in the limit as $t \to 0+$. $\Delta^{\alpha/2}$ is the fractional generalization of the Laplace operator, defined by (2.14) $\hbar$ is the Planck constant defined by (1.4). $N_0(x)$ is an arbitrary function, and $N(x,t)$ is the wave function. Then for the solution of (3.1), subject to the above constraints, there holds the formula

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x - \tau, t) N_0(\tau) d\tau,$$

$$G_1(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{1/\alpha}t^{\mu/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(\mu+\nu(1-\mu),\mu/\alpha),(1,\frac{1}{2})\\(1,1/\alpha),(1,1),(1,\frac{1}{2})\end{matrix}\right],$$

$$(\alpha > 0, \eta = \frac{i\hbar}{2m}), \tag{3.5}$$

where $H_{3,3}^{2,1}(.)$ is the familiar H-function (see [19]).

P r o o f. If we apply the Laplace transform with respect to the time variable t and Fourier transform with respect to space variable x and use the initial conditions (3.2)-(3.4) and the formulas (2.9) and (2.11), then the given equation (3.1) transforms into the form

$$s^{\mu}N^{\sim *}(k,s) - s^{\nu(\mu-1)}N_0(k) = -\eta|k|^{\alpha}N^{\sim *}(k,s), \quad (\eta = \frac{i\hbar}{2m}), \tag{3.6}$$

where according to the convention followed, the symbol ~ will stand for the Laplace transform with respect to time variable t and * represents the Fourier transform with respect to space variable x. Solving for $N^{\sim *}(k,s)$, it yields

$$N^{\sim^*}(k,s) = \frac{N_0(k)s^{\nu(\mu-1)}}{s^{\mu}+\eta|k|^{\alpha}}. \tag{3.7}$$

On taking the inverse Laplace transform of (3.7) by means of the formula

$$L^{-1}\{\frac{s^{\beta-1}}{a+s^{\alpha}}\} = t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-at^{\alpha}), \tag{3.8}$$

where $\Re(s) > 0, \Re(\alpha) > 0, \Re(\alpha-\beta) > -1$, it is seen that

$$N^*(k,t) = N_0(k)t^{\mu+\nu(1-\mu)-1}E_{\mu,\mu+\nu(1-\mu)}(-\eta t^{\mu}|k|^{\alpha}), \tag{3.9}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function [7]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \quad (\alpha,\beta\in\mathbb{C}; \min\{\Re(\alpha),\Re(\beta)\} > 0). \tag{3.10}$$

Taking the inverse Fourier transform of (3.9), we find that

$$N(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{2\pi}\int_{-\infty}^{\infty}N_0(k)E_{\mu,\mu+\nu(1-\mu),}(-\eta t^{\mu}|k|^{\alpha})\exp(-ikx)dk,$$

$$(\eta = \frac{i\hbar}{2m}). \tag{3.11}$$

If we now apply the convolution theorem of the Fourier transform to (3.11) and make use of the following inverse Fourier transform formula given by Haubold et al. [10]:

$$F^{-1}\left[E_{\beta,\gamma}(-at^{\beta}|k|^{\alpha});x\right]=\frac{1}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{a^{1/\alpha}t^{\beta/\alpha}}\left|\begin{matrix}(1,1/\alpha),(\gamma,\beta/\alpha),(1,\frac{1}{2})\\(1,1/\alpha),(1,1),(1,\frac{1}{2})\end{matrix}\right.\right], \tag{3.12}$$

where $\min\{\Re(\alpha),\Re(\beta)>0,\Re(\gamma)\}>0,\alpha>0$, it gives the solution in the form

$$N(x,t)=\int_{-\infty}^{\infty}G_1(x-\tau,t)N_0(\tau)d\tau,$$

where

$$G_1(x,t)=\frac{t^{\mu+\nu(1-\mu)-1}}{2\pi}\int_{-\infty}^{\infty}\exp(-ikx)E_{\mu,\mu+\nu(1-\mu)}(-\eta t^{\mu}|k|^{\alpha})dk$$

$$=\frac{t^{\mu+\nu(1-\mu)-1}}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{1/\alpha}t^{\mu/\alpha}}\left|\begin{matrix}(1,1/\alpha),(\mu+\nu(1-\mu),\mu/\alpha),(1,\frac{1}{2})\\(1,1/\alpha),(1,1),(1,\frac{1}{2})\end{matrix}\right.\right],\ (\alpha>0,\eta=\frac{i\hbar}{2m}), \tag{3.13}$$

where $\min\{\Re(\alpha),\Re(\mu),\Re(\mu+\nu(1-\mu)\}>0,\alpha>0$.

This completes the proof of the theorem. ∎

4. Special cases

If we set $\nu=0$, then the Hilfer fractional derivative (2.10) reduces to a Riemann-Liouville fractional derivative (2.6), and the theorem yields the following:

Corollary 1.1. *Consider the following one dimensional space-time fractional Schrödinger equation of a free particle of mass m, defined by*

$${}_0^{RL}D_t^{\mu}N(x,t)=(\frac{i\hbar}{2m})\Delta^{\alpha/2}N(x,t),\ \ -\infty<x<\infty;\ t>0,\ 0<\alpha\leq 2, \tag{4.1}$$

with the initial conditions

$${}_0^{RL}D_t^{(\mu-1)}N(x,0)=N_0(x);\ [{}_0^{RL}D_t^{(\mu-2)}N(x,0)]=0,\ \ -\infty<x<\infty,1<\mu\leq 2, \tag{4.2}$$

and

$$\lim_{|x|\to\infty}N(x,t)=0, \tag{4.3}$$

where ${}_{0}^{RL}D_t^{\mu}$ *is the Riemann-Liouville fractional derivative operator of order* μ *defined by (2.6),* $[{}_0^{RL}D_t^{(\mu-1)}N(x,0)]$ *means the Riemann-Liouville fractional partial derivative of* $N(x,t)$ *with respect to* t *of order* $(\mu-1)$ *evaluated at* $t=0$*. Similarly* $[{}_0D_t^{(\mu-2)}N(x,0)]$ *means the Riemann-Liouville fractional partial derivative of* $N(x,t)$ *with respect to* t *of order* $(\mu-2)$ *evaluated at* $t=0$, $\Delta^{\alpha/2}$ *is the fractional generalization of the Laplace operator defined by (2.14),* $\hbar$ *is the Planck constant* $N_0(x)$ *is an arbitrary function, and* $N(x,t)$ *is the wave function. Then for the solution of (4.1), subject to the above constraints, there holds the formula*

$$N(x,t) = \int_{-\infty}^{\infty} G_2(x-\tau,t)N_0(\tau)d\tau,$$

where

$$G_2(x,t) = \frac{t^{\mu-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{1/\alpha}t^{\mu/\alpha}}\left|_{(1,1/\alpha),(1,1),(1,\frac{1}{2})}^{(1,1/\alpha),(\mu,\mu/\alpha),(1,\frac{1}{2})}\right.\right],$$

$$(\alpha > 0, \eta = \frac{i\hbar}{2m}). \tag{4.4}$$

When $\nu = 1$, then the Hilfer fractional space derivative (2.10) reduces to a Caputo fractional derivative operator (2.7), and it yields the following result obtained by the authors in a slightly different form [29]:

Corollary 1.2. *Consider the following one dimensional space-time fractional Schrödinger equation of a free particle of mass* m*, defined by*

$${}_0^cD_t^{\mu}N(x,t) = \frac{i\hbar}{2m}\Delta^{\alpha/2}N(x,t),\ 0<\mu\le 1, 0<\alpha\le 2, -\infty<x<\infty, t>0, \tag{4.5}$$

with initial conditions

$$N(x,0+) = N_0(x), \quad -\infty < x < \infty, \tag{4.6}$$

and

$$\lim_{|x|\to\infty} N(x,t) = 0, \tag{4.7}$$

where ${}_0^cD_t^{\mu}$ *is the Caputo fractional derivative operator, defined by (2.7),* $\Delta^{\alpha/2}$ *is the fractional generalization of the Laplace operator, defined by (2.14),* $\hbar$ *is the Planck constant defined by (1.4),* $N_0(x)$ *is an arbitrary function, and* $N(x,t)$ *is the wave function. Then for the solution of (4.5), subject to the above constraints, there holds the formula*

$$N(x,t) = \int_{-\infty}^{\infty} G_3(x-\tau,t)N_0(\tau)d\tau,$$

where

$$G_3(x,t) = \frac{t}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{1/\alpha}t^{\mu/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(1,\mu/\alpha),(1,\frac{1}{2})\\(1,1/\alpha),(1,1),(1,\frac{1}{2})\end{matrix}\right] \quad (\alpha>0, \eta=\frac{i\hbar}{2m}). \quad (4.8)$$

For $N_0(x) = \delta(x)$, where $\delta(x)$ is the Dirac-delta function, Theorem 1 yields the following:

Corollary 1.3. *Consider the following one dimensional space-time fractional Schrödinger equation of a free particle of mass m, defined by*

$$ {}_0D_{0+}^{\mu,\nu}N(x,t) = (\frac{i\hbar}{2m})\Delta^{\alpha/2}N(x,t); \quad -\infty < x < \infty, t>0, 0<\alpha\le 2, \quad (4.9)$$

with the initial condition

$$\left(I_{0+}^{(1-\nu)(\mu-1),0}\right) N(x,0+) = \delta(x), \; -\infty<x<\infty, \; 0<\mu<1, 0\le\nu\le 1, \quad (4.10)$$

and

$$\lim_{|x|\to\infty} N(x,t) = 0, \quad (4.11)$$

where ${}_0D_{0+}^{\mu,\nu}$ is the Hilfer fractional derivative, defined by (2.11), $\Delta^{\alpha/2}$ is the fractional generalization of Laplace operator defined by (2.14), $\hbar$ is the Planck constant defined by (1.4), and $\delta(x)$ is the Dirac-delta function. Then for the fundamental solution of (4.9) with the initial conditions (4.10)-(4.11) there holds the formula

$$N(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{1/\alpha}t^{\mu/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(\mu+\nu(1-\mu),\mu/\alpha),(1,\frac{1}{2})\\(1,1/\alpha),(1,1),(1,\frac{1}{2})\end{matrix}\right],$$

$$(\alpha>0; \eta=\frac{i\hbar}{2m}). \quad (4.12)$$

Note 3. We note that for $\alpha=2, \nu=1$, (4.12) gives

$$N(x,t) = \frac{1}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{(\eta t^{\alpha})^{1/2}}\middle|\begin{matrix}(1,\mu/2)\\(1,1)\end{matrix}\right], \quad (\alpha>0, \eta=\frac{i\hbar}{2m}), \quad (4.13)$$

which is the explicit form of the solution discussed by Debnath [4, p.152].

Note 4. It is interesting to note that if we set $\alpha = 2$. $\nu=1, \mu=1$ in (3.5), then the classical Gaussian density [5, p.118] is recovered:

$$\frac{1}{\sqrt{4\pi a t}}\exp\left(-\frac{x^2}{4at}\right). \quad (4.14)$$

5. Computational representations of the fundamental solution (4.12)

In this section, we will derive the computational representation of the fundamental solution (4.12), which can be expressed in terms of the Mellin-Barnes type integral with the help of the definition of the H-function (2.18) as

$$N(x,t) = \frac{1}{\pi x}\frac{t^{\mu+\nu(1-\mu)-1}}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(1-s\alpha)}{\Gamma(\mu+\nu(1-\mu)-s\mu)}\sin\left[\frac{\pi s\alpha}{2}\right]\left[\frac{x^\alpha}{\eta t^\mu}\right]^s ds, \tag{5.1}$$

where $L = L_{i\tau\infty}$ $(\tau \in \mathbb{R})$ is an infinite contour, which extends from $\tau - i\infty$ to $\tau + i\infty$, and separates all the poles of $\Gamma(1-s)$ at the points $s = 1+n$ $(n \in \mathbb{N}_0)$ and $\Gamma(1-s\alpha)$ at the points $s = \frac{1+n}{\alpha}(\alpha > 0l, n \in \mathbb{N}_0)$ to the left and all the poles of $\Gamma(s)$ at the points $s = -n$ $(n \in \mathbb{N}_0)$ to the right of it.

Let us assume that the poles of the gamma functions in the integrand of (5.1) are all simple. Now evaluating the sum of residues in ascending powers of x^α by calculating the residues at the poles of $\Gamma(1-s)$ at the points $s = 1+n$, $(n \in \mathbb{N}_0)$ and $\Gamma(1-s\alpha)$, at the points $s = (1+n)/\alpha, (n \in \mathbb{N}_0)$, we obtain the following representation of the fundamental solution (4.12) in terms of two convergent series in ascending powers of x^α:

$$N(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\pi x}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(1-\alpha n)}{\Gamma(\mu+\nu(1-\mu-n\mu)}\sin\left[\frac{n\pi\alpha}{2}\right]\left[\frac{x^\alpha}{\eta t^\mu}\right]^n$$

$$+\frac{t^{\mu+\nu(1-\mu)-1}}{\pi x\alpha}\sum_{n=1}^{\infty}\frac{\Gamma(1-n/\alpha)\Gamma(1+n/\alpha)}{n!\Gamma(\mu+\nu(1-\mu)-n\mu/\alpha)}\left[\frac{-x^\alpha}{\eta t^\mu}\right]^{n/\alpha}, \quad (0 < x < 1), \tag{5.2}$$

where $\left\{\left|\frac{x^\alpha}{\eta t^\mu}\right|\right\} < 1, \eta = \frac{i\hbar}{2m}, \alpha > 0$.

Finally, if we calculate the residues at the poles of $\Gamma(s)$ of the integrand of (4.1) at the points $s = -n$, $(n \in \mathbb{N}_0)$, it gives

$$N(x,t) = \frac{t^{\mu+\nu(1-\mu)-1}}{\pi x}\sum_{n=0}^{\infty}\frac{\Gamma(1+\alpha n)}{\Gamma(\mu+\nu(1-\mu)+n\mu)}\sin\left[\frac{-n\pi\alpha}{2}\right]\left[\frac{-x^\alpha}{\eta t^\mu}\right]^{-n}$$

$$(0 < x < \infty), \tag{5.3}$$

where $\left\{\left|\frac{x^\alpha}{\eta t^\mu}\right|\right\} > 1, \eta = \frac{i\hbar}{2m}, \alpha > 0.$

References

[1] M. Bhatti, Fractional Schrödinger wave and fractional uncertainty principle. *Int. J. Contemp. Math. Sciences* **2**, No 19 (2007), 943-950.

[2] D. Brockmann and I.M. Sokolov, Lévy flights in external force fields:from models to equations. *Chemical Physics* **284** (2002), 409-421.

[3] M. Caputo, *Elasticitá e Dissipazione.* Zanichelli, Bologna, 1969.

[4] L. Debnath, Fractional integral and fractional differential equations in fluid mechanics. *Fract. Calc. Appl. Anal.* **6** (2003), 119-155.

[5] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers,* Second Edition. Birkhäuser, Boston, Basel, Berlin, 2005.

[6] L. Debnath and M. Bhatti, On fractional Schrödinger and Dirac equations. *Intern. J. Pure Appl. Math.* **15** (2004), 1-11.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi (Eds), *Higher Transcendental Functions*, Vol. **3**. McGraw-Hill, New York, 1955.

[8] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals.* McGraw-Hill, New York, 1965.

[9] X. Guo and M. Xu, Some physical applications of Schrödinger equation. *J. Math. Phys.* **47082104** (2006), doi: 10. 1063/1. 2235026 (9 pages).

[10] H.J. Haubold, A.M. Mathai and R.K. Saxena, Solution of reaction-diffusion equations in terms of the H-function, *Bull. Astro. Soc.India* **35**, No 4 (2007), 681-689.

[11] B.I. Henry and S.L. Wearne, Fractional reaction-diffusion. *Physica A* **276** (2000), 448-455.

[12] R. Hilfer, Fractional time evolution. In: R. Hilfer (Editor), *Applications of Fractional Calculus in Physics*, World Scientific Publishing Company, Singapore-New Jersey-London-Hong Kong, 2000, 87-130.

[13] R. Hilfer, On fractional relaxation. *Fractals* **11** (2003), 251-257.

[14] R. Hilfer, Threefold Introduction to Fractional Derivatives, Anomalous Transport: Foundations & Applications (Ed. by Rainer Klages, Günter Radons, and Igor M. Sokolov), Wiley-VCH Verlag, GmbH & Co.KCaA, Weinheim, 2009.

[15] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* Elsevier, Amsterdam, 2006.

[16] N. Laskin, Fractals and quantum mechanics. *Chaos* **10**, No 4 (2000), 780-790.

[17] N. Laskin, Fractional quantum mechanics and Levy path integrals. *Physics Letters A* **268** (2000), 298-305.

[18] N. Laskin, Fractional Schrödinger equation. *Physical Review E* **66**, 056108, arXiv: quant-ph/0206098 v1, 14 Jun 2002.

[19] A.M. Mathai, R.K. Saxena and H.J. Haubold, *The H-function: Theory and Applications*. Springer, New York, 2010.

[20] R. Metzler and J. Klafter, The random walk, a guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **339** (2000), 1-77.

[21] K.S. Miller and B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, 1993.

[22] M. Naber, Distributed order fractional sub-diffusion, *Fractals* **12**, No 1 (2004), 23-32.

[23] K.B. Oldham and J. Spanier, *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, New York, 1974.

[24] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[25] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science Publishing, Switzerland, 1993.

[26] R.K. Saxena, A.M. Mathai and H.J. Haubold, Fractional reaction-diffusion equations. *Astrophysics and Space Science* **305** (2006), 289-296.

[27] R.K. Saxena, A.M. Mathai and H.J. Haubold, Solution of generalized fractional reaction-diffusion equations. *Astrophysics and Space Science* **305** (2006), 305-313.

[28] R.K. Saxena, A.M. Mathai and H.J. Haubold, Reaction-diffusion systems and nonlinear waves. *Astrophysics and Space Science* **305** (2006), 297-303.

[29] R.K. Saxena, R. Saxena and S.L. Kalla, Computational solution of a fractional generalization of the Schrödinger equation occurring in quantum mechanics. *Appl. Math. Comput.* **216** (2010), 1412-1417.

[30] H.M. Srivastava and Z. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. *Appl. Math. Comput.* **211** (2009), 198-210.

[31] A. Tofight, Probability structure of time fractional Schrödinger equation. *Acta Physica Polonica A* **116**, No 2 (2009), 111-118.

[a] *Dept. of Mathematics & Statistics, Jai Narain Vyas University*
Jodhpur - 342005, INDIA
e-mail: ram.saxena@yahoo.com

[b] *Dept. of Civil Engineering, Jai Narain Vyas University*
Jodhpur – 342005, INDIA
e-mail: saxravi@yahoo.co.in

Received: January 4, 2010

[c] *(Corresp. author):*
Institute of Mathematics – VIHE (Vyas Institute of Higher Education)
15-B, Pal-Link Road
Jodhpur – 342008, INDIA
e-mails: shyamkalla@gmail.com , shyamkalla@yahoo.com